Light beams with minimum phase–space product

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We derive a reciprocity inequality involving the product of the effective size of a statistically stationary, planar, secondary source of any state of coherence and of the angular spread of the far-zone intensity generated by the source. We show that of all possible such sources, the fully spatially coherent lowest-order Hermite–Gaussian laser mode has the smallest possible reciprocity product.

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Optical reciprocity inequalities are an important subject in the context of diffraction of spatially *coherent* light. An inequality of this type may be formulated as follows: Consider a collimated, spatially coherent light wave of wavelength λ, incident on an aperture of area A in a plane opaque screen. The illuminated aperture may then be regarded as a secondary source. According to a well-known result of Fraunhofer diffraction theory,[1] the area A of the aperture and the solid angle Ω, into which most of the light from the secondary source propagates, satisfy the reciprocity inequality

$$\Omega A \geq \lambda^2. \tag{1}$$

In general, however, there are not many light sources that are fully spatially coherent; most sources are only *partially* coherent. Studying the properties of partially coherent sources and those of the fields that they generate has attracted growing interest. In particular, it has been shown that some partially coherent sources may generate the same far-zone intensity distribution as is produced by a single-mode laser.[2] In this connection, it is appropriate to formulate an optical reciprocity inequality for sources of *any state of coherence* and the fields that they generate. Some preliminary results concerning this subject were reported in Ref. 3.

In this Letter, we formulate a reciprocity inequality for the phase–space product of the rms radius of a statistically stationary, planar, secondary source of any state of coherence and of the rms angular spread of the radiant intensity distribution that the source generates. We then show, with the help of second-order coherence theory in the space-frequency domain (Ref. 4; see also Sect. 4.7.1 of Ref. 5), that the phase–space product attains minimum for the fully spatially coherent lowest-order Hermit–Gaussian mode, such as is generated by some well-stabilized lasers.

We begin by recalling that the cross-spectral density $W(\boldsymbol{\rho}, \boldsymbol{\rho}', \omega)$ of a partially coherent, planar source can be represented as a Mercer-type series of spatially completely coherent modes $\phi_f(\boldsymbol{\rho}, \omega)$ at given frequency ω, by means of the formula

$$W(\boldsymbol{\rho}, \boldsymbol{\rho}', \omega) = \sum_f \lambda_f \phi_f^*(\boldsymbol{\rho}, \omega)\phi_f(\boldsymbol{\rho}', \omega). \tag{2}$$

Here $\boldsymbol{\rho}$ and $\boldsymbol{\rho}'$ are two-dimensional vectors specifying points in the source plane; the subscript f stands for a set of integers labeling the modes, and λ_f is the eigenvalue corresponding to the mode ϕ_f. The modes can be chosen to form an orthonormal set. Each mode is a solution of the integral equation

$$\int \mathrm{d}^2\rho W(\boldsymbol{\rho}, \boldsymbol{\rho}', \omega)\phi_f(\boldsymbol{\rho}) = \lambda_f \phi_f(\boldsymbol{\rho}'), \tag{3}$$

where the eigenvalues λ_f are necessarily nonnegative,

$$\lambda_f \geq 0. \tag{4}$$

The effective rms radius of the source may be defined as the square root of the expression

$$\langle \rho^2 \rangle = \frac{\int \mathrm{d}^2\rho \boldsymbol{\rho}^2 I(\boldsymbol{\rho})}{\int \mathrm{d}^2\rho I(\boldsymbol{\rho})}, \tag{5}$$

where $I(\boldsymbol{\rho}) = W(\boldsymbol{\rho}, \boldsymbol{\rho}, \omega)$ is the intensity of the source at the point $\boldsymbol{\rho}$ at frequency ω. It follows from Eqs. (2)and (5) that

$$\langle \rho^2 \rangle = \frac{\sum_f \lambda_f \int \mathrm{d}^2\rho \rho^2 |\phi_f(\boldsymbol{\rho})|^2}{\sum_f \lambda_f}. \tag{6}$$

Similarly, we define the rms angular spread of the radiant intensity $J(\mathbf{s}_\perp)$ (Sect. 5.2.1 of Ref. 5) of the field generated by the source as the square root of the expression

$$\langle s_\perp^2 \rangle = \frac{\int \mathrm{d}^2 s_\perp \mathbf{s}_\perp^2 J(k\mathbf{s}_\perp, \omega)}{\int \mathrm{d}^2 s_\perp J(k\mathbf{s}_\perp, \omega)}. \tag{7}$$

Here $k = \omega/c$ and $\mathbf{s}_\perp$ is a two-dimensional vector representing the projection onto the source plane of a unit three-dimensional vector $\hat{\mathbf{s}}$, pointing from the source to the far zone (see Fig. 1). It should be noted that $|\mathbf{s}_\perp| = \sin\theta$, where θ is the angle between the "propagation direction" and the normal to the source plane. If the source generates a paraxial beam, then $|\mathbf{s}_\perp| \approx \theta$. The radiant intensity may be expressed in terms of the cross-spectral density of the source by the formula (Sect. 5.2.1 of Ref. 5)

$$J(k\mathbf{s}_\perp, \omega) = \int \frac{\mathrm{d}^2\rho}{2\pi} \int \frac{\mathrm{d}^2\rho'}{2\pi} W(\boldsymbol{\rho}, \boldsymbol{\rho}', \omega) \times \exp[-ik\mathbf{s}_\perp \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}')]. \tag{8}$$

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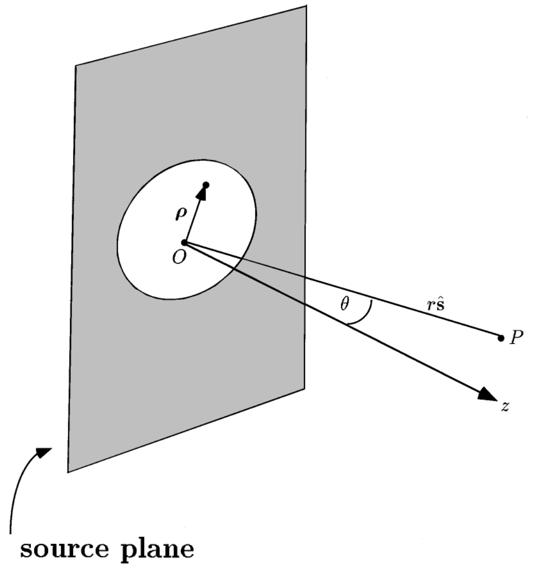

Fig. 1. Geometry of a partially coherent, statistically stationary, planar, secondary source. P is a point in the far zone of the source occupying a portion of the plane $z = 0$. $\overline{OP} = r\hat{\mathbf{s}}$, $(\hat{\mathbf{s}}^2 = 1)$. θ denotes the angle which the line $\overline{OP}$ makes with the z axis.

On substituting for $W(\boldsymbol{\rho}, \boldsymbol{\rho}', \omega)$ from Eq. (2) into Eq. (8) and using the resulting expression in Eq. (7), we obtain for the square of the rms angular spread of the radiant intensity the formula

$$\langle s_\perp^2 \rangle = \frac{\sum_f \lambda_f \int \mathrm{d}^2(ks_\perp) s_\perp^2 |\tilde{\phi}_f(k\mathbf{s}_\perp)|^2}{\sum_f \lambda_f}, \tag{9}$$

where $\tilde{\phi}_f(k\mathbf{s}_\perp)$ is the Fourier transform of the mode function $\phi_f(\boldsymbol{\rho})$, defined as

$$\tilde{\phi}_f(k\mathbf{s}_\perp) = \int \frac{\mathrm{d}^2\rho}{2\pi} \phi_f(\boldsymbol{\rho}) \exp(-ik\mathbf{s}_\perp \boldsymbol{\rho}). \tag{10}$$

It can be seen from Eqs. (6) and (9) that the effective rms radius of the source and the rms angular spread of the radiant intensity of the field generated by the source are just weighted superpositions of the average effective size and the effective angular spread of each mode. The mode functions $\phi_f(\boldsymbol{\rho})$ and their Fourier transforms $\tilde{\phi}_f(k\mathbf{s}_\perp)$ are formally analogous to the quantum-mechanical wave functions in coordinate and in momentum representations, respectively. This fact allows us to utilize the approach developed in the context of quantum mechanics[6] to derive the reciprocity relation for the phase–space product of conjugate variables. For this purpose, we consider the nonnegative quantity

$$\Lambda(\alpha) = \frac{\sum_f \lambda_f \int \mathrm{d}^2\rho (\mathbf{g}_f^*, \mathbf{g}_f)}{\sum_f \lambda_f} \geq 0, \tag{11}$$

where the vector $\mathbf{g}_f$ is defined as

$$\mathbf{g}_f = \boldsymbol{\rho}\phi_f(\boldsymbol{\rho}) + \alpha \nabla \phi_f(\boldsymbol{\rho}), \tag{12}$$

with α being an arbitrary real constant, *independent of* the subscript f. On substituting from Eq. (12) into Eq. (11), we obtain for $\Lambda(\alpha)$ the expression

$$\Lambda(\alpha) = \frac{\sum_f \lambda_f \int \mathrm{d}^2\rho [\rho^2 |\phi_f|^2 + \alpha \boldsymbol{\rho} \cdot \nabla |\phi_f|^2}{\sum_f \lambda_f} + \frac{\sum_f \lambda_f \int \mathrm{d}^2\rho \alpha^2 \nabla\phi_f \nabla\phi_f^*]}{\sum_f \lambda_f} \geq 0. \tag{13}$$

Next, we use the properties of Fourier transforms to rewrite the last term inside the sum on the right-hand side of Eq. (13) in the form

$$\int \mathrm{d}^2\rho \nabla\phi_f \nabla\phi_f^* = \int \mathrm{d}^2(ks_\perp) k^2 s_\perp^2 |\tilde{\phi}_f(k\mathbf{s}_\perp)|^2, \tag{14}$$

where contributions which give rise to evanescent waves ($|\mathbf{s}_\perp| > 1$) have been omitted. On integrating by parts the second term on the right-hand side of Eq. (13) and on substituting Eq. (14) into Eq. (13), we obtain the inequality

$$\Lambda(\alpha) = \langle \rho^2 \rangle - 2\alpha + \alpha^2 k^2 \langle s_\perp^2 \rangle \geq 0, \tag{15}$$

where we have used the definitions of the rms radius (6) of the source and that of the rms angular spread (9). In order for the inequality (15) to hold for any value of α, the square of the rms radius of the source and the square of the rms angular spread of the radiant intensity must satisfy the inequality

$$\langle \rho^2 \rangle \langle s_\perp^2 \rangle \geq \left(\frac{\lambda}{2\pi}\right)^2, \tag{16}$$

where $\lambda = 2\pi/k$ is the wavelength. If the source generates a paraxial beam, then, since $|\mathbf{s}_\perp| \approx \theta$, Eq. (16) becomes

$$\langle \rho^2 \rangle \langle \theta^2 \rangle \geq \left(\frac{\lambda}{2\pi}\right)^2. \tag{17}$$

The inequality (17) is similar to the reciprocity inequality (1) for a phase–space product of the fields generated by coherent sources.

We will now determine the class of sources for which the phase–space product attains minimum. We observe that since the quantity $\Lambda(\alpha)$ in Eq. (11) is nonnegative, its minimum is equal to zero, which in turn is achieved if and only if $\mathbf{g}_f = 0$ for all f. According to Eq. (12) this condition is equivalent to the set of equations

$$\boldsymbol{\rho}\phi_f(\boldsymbol{\rho}) + \alpha \nabla \phi_f(\boldsymbol{\rho}) = 0 \tag{18}$$

for $\phi_f(\boldsymbol{\rho})$. The normalized solutions of these equations are

$$\phi_f(\boldsymbol{\rho}) = \frac{1}{2\pi\sigma^2} \exp(-\rho^2/2\sigma^2), \tag{19}$$

where $\sigma = \alpha^{1/2}$ is a rms radius, which is the same for each mode. Finally, on substituting Eq. (19) into Eq. (2), we obtain for the cross-spectral density $W_0(\boldsymbol{\rho}, \boldsymbol{\rho}', \omega)$ of the source which has minimum phase–space product, the expression

$$W_0(\boldsymbol{\rho}, \boldsymbol{\rho}', \omega) = A \exp(-\rho^2/2\sigma^2) \exp(-\rho'^2/2\sigma^2), \tag{20}$$

where $A = \sum_f \lambda_f / 2\pi\sigma^2$. It follows at once from Eq. (20) that (i) the source for which the phase–space product attains its minimum is fully spatially coherent[7] and (ii) its cross-spectral density is just a product of the lowest-order Hermite–Gaussian functions, which corresponds to the fully spatially coherent Gaussian Shell-model source (Sect. 5.4.2 of Ref. 5). We have thus established the following theorem:

Among all statistically stationary, planar, secondary sources, the one that gives rise to the minimum value of the phase–space product,

$$\Pi \equiv \langle \rho^2 \rangle \langle \mathbf{s}_\perp^2 \rangle, \qquad (21)$$

where $\langle \rho^2 \rangle$ *is the square of the rms radius of the source and* $\langle \mathbf{s}_\perp^2 \rangle = \langle \sin^2 \theta \rangle$ *is the square of the rms angular spread of the radiant intensity produced by the source, is the source that generates the lowest-order Hermite–Gaussian field.* Such field distribution is produced, for instance, by some well-stabilized single-mode lasers.

To summarize, we have derived an inequality that is satisfied by the phase–space product of the rms radius of a statistically stationary, planar, secondary source of any state of coherence and of the rms angular spread of the radiant intensity distribution that this source generates. We have also shown that the phase–space product attains minimum for the fully spatially coherent lowest-order Hermite–Gaussian distribution.

The theorem that we just established has an interesting implication for Gaussian Shell-model sources of any state of coherence. On one hand, it is known[2] that one can trade off the rms radius of the Gaussian Shell-model source for the rms angular spread of the radiant intensity distribution generated by the source. On the other hand, it follows from our theorem that the requirement for the phase–space product to attain its minimum makes the fully coherent Gaussian Shell-model source genuinely unique.

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7. This is because for the field distribution (20) the modulus of the spectral degree of coherence $|\mu_0(\boldsymbol{\rho}, \boldsymbol{\rho}', \omega)|$, defined as

$$|\mu_0(\boldsymbol{\rho}, \boldsymbol{\rho}', \omega)| \equiv \left| \frac{W_0(\boldsymbol{\rho}, \boldsymbol{\rho}', \omega)}{\sqrt{W_0(\boldsymbol{\rho}, \boldsymbol{\rho}, \omega)}\sqrt{W_0(\boldsymbol{\rho}', \boldsymbol{\rho}', \omega)}} \right|,$$

is equal to unity.